Problems for the 8th Annual Math Match 2023

1. Create a 4-digit number with different digits taken from the set $\{1, 2, \ldots , 9\}$. Then, find the sum of all such 4-digit numbers.

2.

If is a diagonal of the polygon, then choose the farthest (or one of the farthest) vertex and on each side of this diagonal. Let and be the two lines parallel to , passing corespondingly through and . Let be the rectangle formed by lines , , , and . All vertices of the polygon must lie within this rectangle, by construction.

We have the following result $= 2 \cdot$ $\leq 2 \cdot$

If is one of the sides of the polygon, one of the lines or are the same as and the above reasoning holds as well.

5. First, notice that the area of the octagon from our problem is equal to the area of a cyclic octagon with alternating sides of length 1 and 2. By symmetry, all the interior angles are the same and their sum equals $(8-2)180° = 1080°$. Therefore, each interior angle is equal to $\frac{1080°}{8} = 135°$, and each exterior angle is equal to $45°$.

Consider extension of the octagon to the quadrilateral , as on the diagram. Since $=$ $= 45°$ then $= 90°$. Similarly, angles at , , are all $90°$. So the quadrilateral is a square.

If $= 1$, then $= \frac{1}{\sqrt{2}}$. Similarly, all the arms of the right angle triangles are equal to $\frac{1}{\sqrt{2}}$. The area of the octagon equals to the area of the square minus the area of the 4 right angle triangles. So, we have $2 + \frac{2}{\sqrt{2}}\ {}^2 - 2\ \frac{1}{\sqrt{2}}\ {}^2 = \ 2 + \sqrt{2}\ {}^2 - 1 = \ + \ \sqrt{}$

6. Group $1!\,2! \cdot 3!\,4! \cdot \ldots \cdot 99!\,100! = (1!\,3! \ldots 99!)^2 \cdot 2^{50} \cdot 50!$ If we remove **50!**, the remaining product is a perfect square of $1!\,3! \ldots 99! \cdot 2^{25}$.

7. If $= 3$ then $= (\ )^3$, hence is a perfect cube. There are 100 natural numbers satisfying this condition.
If $= 3 + 1$, then $= (\ )^3$, which is a perfect cube if is a perfect cube. If $= 3 + 2$, then $= (\ )^3\ {}^2$, which is a perfect cube if is a perfect cube.
For $1 \leq \ \leq 300$ there are four numbers of the form 3k +1 or 3k +2 which are perfect cubes (1, $8 = 2^3$, 64 = 4^3, and 125 =5^3). So, together there are **104** integers having the requested properties.

8. Using the given equation for $= 0$, and $= 2$, we obtain $(0) = 0$ and $(1) = 0$.

Therefore, $(\ )$ must have a form $(\ --($